Section 12.1

Parametric Equations

Suppose a boat travels across a still lake. Recording the boats movements, a time t will correspond to a position on the lake, and under for some choice of coordinate axes a point (x, y). As both x and y depend on t, we write $(x(t), y(t))$. If the boat begins at a point a units from the origin on the positive x-axis and travels in a perfect circle around the origin counter-clockwise at a radius a and with a fixed speed, we have $x(t) = a\cos\left(\frac{2\pi t}{T}\right), y(t) = a\sin\left(\frac{2\pi t}{T}\right)$ where T is the time it takes to complete one full revolution.

For any function $f(x)$, points on the graph of f have coordinates $(x, f(x))$. We can then parametrize this curve by $x(t) = t, y(t) = f(t)$ which simply gives the points $(t, f(t))$. We can attempt to work backwards if a curve is a function. *N.B. if the curve is differentiable at a point then it is locally given by a 1-to-1 function.*

Example 12.1.1. Find the curve of the parametric equations $x(t) = 2t, y(t) = \frac{1}{2}t^2 - 4$.

Definition 12.1.2. Positive Orientation

The direction in which a parametric curve is generated as the parameter increases is called the *positive orientation* of the curve. We indicate this with arrows on the curve.

Example 12.1.3. Confirm that the parametric equations

$$x = 4\sin 2\pi t, y = 4\cos 2\pi t, \text{ for } 0 \leq t \leq 1$$

describe a circle of radius 4. Indicate the equations orientation.

Example 12.1.4. Find a parametric equation of a line passing through the point (x_0, y_0) with slope m.

Example 12.1.5. Describe the line segment given by the equations

$$x = -2 + 3t, y = 4 - 6t, \text{ for } -1 \leq t \leq 1$$

Theorem 12.1.6. Derivative of Parametric Curves

Let $x = f(t)$ and $y = g(t)$, where f and g are differentiable on an interval $[a, b]$. Then the slope of the line tangent to the curve at the point corresponding to t is

$$\frac{dy}{dx} = \frac{g'(t)}{f'(t)}$$

provided $f'(t) \neq 0$.

Example 12.1.7. Find the horizontal and vertical tangent lines to the curve $x = 4\cos t, y = 16\sin t$ for $0 \leq t \leq 2\pi$.

The same derivation from chapter 7 will compute the arc length of a parametric curve

Theorem 12.1.8. Arc Length

Consider the curve described by the parametric equations $x = f(t), y = g(t)$, where f' and g' are continuous, and the curve is traversed once for $a \leq t \leq b$. The *arc length* of the curve between $(f(a), g(a))$ and $(f(b), g(b))$ is

$$L = \int_a^b \sqrt{f'(t)^2 + g'(t)^2} dt.$$

Example 12.1.9. Prove that the circumference of a circle with radius r is $2\pi r$.

Example 12.1.10.

1. Parametrize the curve $x = (y-5)^2 + \sqrt{y}$.

2. Parametrize the piecewise linear path connecting the points $P(-2,0), Q(0,3), R(4,0)$ in that order where the parameter varies over the interval $0 \leq t \leq 2$.

3. A *cycloid* is the path of a point on a wheel of radius r rolling on a flat surface. The parametric equations for the cycloid are

$$x = a(t - \sin t), y = a(1 - \cos t)$$

for $t \geq 0$. On what interval is the first arch of the cycloid? Is the arch a segment of a circle?

4. An *astroid* is the path of a point on a circle of radius $a/4$ that rolls inside a circle of radius a. The parametric equations are

$$x = a\cos^3 t, y = a\sin^3 t$$

for $0 \leq t \leq 2\pi$. Find the arc length when $a = 1$.

5. Graph the curve $x = t + \cos 2t, y = t - \sin 4t$ for $-1 \leq t \leq 6$.